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Linearizability conditions of a time-reversible quartic-like system

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ABSTRACT

In this paper we study the linearizability problem of polynomial-like complex differential systems. We give a reduction of linearizability problem of such non-polynomial systems to the problem of polynomial systems. Applying this reduction, we find some linearizability conditions for a time-reversible quartic-like complex system and derive from them conditions of isochronous center for the corresponding real system.

1. Introduction

Motivated by the isochronous center problem of real systems, increasing interests have been attracted to the analytic linearization of planar analytic complex systems. The so-called *isochronous center problem* of real systems is: Is the origin an isochronous center of the real system

$$\begin{cases} \dot{x} = -y + F(x, y), \\ \dot{y} = x + G(x, y), \end{cases} \tag{1.1}$$

where $x, y \in \mathbb{R}$, $\dot{x}$ and $\dot{y}$ denote dx/dt and dy/dt, respectively, and $F(x, y) = O(|(x, y)|^2)$, $G(x, y) = O(|(x, y)|^2)$? In the sense of normal forms, by the Poincaré–Lyapunov Theorem [2,23,28], it is equivalent to determine whether system (1.1) can be linearized to $\dot{u} = -v$, $\dot{v} = u$ with an analytic transformation $u = x + O(|(x, y)|^2)$, $v = y + O(|(x, y)|^2)$ in a neighborhood of the origin, which can be reduced to the linearizability problem of a complex system by the change of variables $Z = x + \mathrm{i}y$, $W = x - \mathrm{i}y$, $T = \mathrm{i}t$. Clearly, $Z = 0$ if and only if $W = 0$ because the conjugacy in real systems implies that $x = y = 0$. More generally, one can consider a planar analytic complex system

$$\begin{cases} \dot{Z} = Z + P(Z, W) = \widetilde{P}(Z, W), \\ \dot{W} = -W + Q(Z, W) = \widetilde{Q}(Z, W) \end{cases} \tag{1.2}$$

in the region

$$\Xi := \{(Z, W) \in \mathbb{C}^2\colon ZW \neq 0\} \cup \{(0, 0)\},$$

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where $\dot{Z}$ and $\dot{W}$ denote dZ/dT and dW/dT respectively and $P(Z,W)=O(|(Z,W)|^2)$, $Q(Z,W)=O(|(Z,W)|^2)$. As concerned in [2,11,28], the *linearizability problem* is: Can system (1.2) be linearized to $\dot{X}=X$, $\dot{Y}=-Y$ by an analytic near-identity transformation

$$X=Z+O\big(\big|(Z,W)\big|^2\big),\qquad Y=W+O\big(\big|(Z,W)\big|^2\big) \tag{1.3}$$

in a neighborhood of the origin?

Most results about analytic linearizability for system (1.2) are obtained for polynomial systems, that is, both P and Q are polynomials. For instance, the problem was solved in [5,6] when P and Q are both homogeneous polynomials of degree 2 or 3 and in [27] when they are both of degree 5. Recently, along with the increasing attention (see e.g. [4,9,10,12,13,15, 24] for results on integrability and attractors) to time-reversible systems, all linearizability conditions for system (1.2) with time-reversibility were obtained in [3] when P and Q are both general cubic polynomials and in [4] when they are both quartic homogeneous polynomials.

Concerning real systems, all isochronous center conditions are obtained for system (1.1) with F and G both being quadratic [22], cubic [25] and quintic homogeneous polynomials [27]. Some results on isochronous centers of time-reversible systems are obtained in [1,3,4]. Since the New Millennium, attentions were also paid to the polynomial-like system

$$\begin{cases}\dot{x}=-y+\big(x^2+y^2\big)^d F_n(x,y),\\ \dot{y}=x+\big(x^2+y^2\big)^d G_n(x,y),\end{cases} \tag{1.4}$$

where d is a real number (may not be an integer) and both $F_n(x,y)$ and $G_n(x,y)$ are homogeneous polynomials of degree n for isochronicity. The center problem and limit cycle bifurcations were discussed in [16] for $n=2$. For real $d\geqslant 0$, conditions of isochronous center are obtained for a special form of (1.4) as $n=4$ in [14]. For integer $d\geqslant 0$, conditions of isochronous center are obtained for system (1.4) as $n=2$ and $n=3$ in [19] and [20] respectively and for a special form of (1.4) with $n=4$ in [21]. Readers can find more results about isochronous centers from [23,28], references therein and the survey paper [2]. As done as above, with the change of variables $Z=x+\mathrm{i}y$, $\overline{Z}=x-\mathrm{i}y$, $T=\mathrm{i}t$, the real system (1.4) can be transformed into the complex system

$$\begin{cases}\dot{Z}=Z+(Z\overline{Z})^d P_n(Z,\overline{Z}),\\ \dot{\overline{Z}}=-\overline{Z}-(Z\overline{Z})^d \overline{P_n(Z,\overline{Z})},\end{cases} \tag{1.5}$$

where $d\geqslant 0$ is a real number, n is an integer, $2d+n>1$ and P_n is a homogeneous polynomial of degree n. Clearly, $\overline{P_n(Z,\overline{Z})}$ is also a homogeneous polynomial of degree n in Z and $\overline{Z}$. For the conjugacy, either $Z\overline{Z}\neq 0$ or $(Z,\overline{Z})=(0,0)$.

In this paper we consider the polynomial-like complex differential system

$$\begin{cases}\dot{Z}=Z+(ZW)^d P_n(Z,W),\\ \dot{W}=-W-(ZW)^d Q_n(Z,W),\end{cases} \tag{1.6}$$

a generalization of (1.5), in the region Ξ (defined just after (1.2)), where Q_n is a homogeneous polynomial of degree n, and discuss on its analytic linearizability problem. We give a reduction of the linearizability problem to the problem of a polynomial one so as to construct a near-identity transformation

$$X=Z+o\big(\big|(Z,W)\big|\big),\qquad Y=W+o\big(\big|(Z,W)\big|\big), \tag{1.7}$$

which is analytic near the origin in the region Ξ and linearize (1.6). We apply the reduction to the quartic-like complex system

$$\begin{cases}\dot{Z}=Z+(ZW)^d\big(a_{31}Z^3W+a_{22}Z^2W^2+a_{13}ZW^3+a_{04}W^4\big),\\ \dot{W}=-W-(ZW)^d\big(b_{04}Z^4+b_{13}Z^3W+b_{22}Z^2W^2+b_{31}ZW^3\big)\end{cases} \tag{1.8}$$

with the time-reversibility condition

$$a_{13}^5b_{04}^3-a_{04}^3b_{13}^5=a_{22}^3b_{13}-a_{13}b_{22}^3=a_{22}^5b_{04}-a_{04}b_{22}^5=a_{22}a_{31}-b_{22}b_{31}=a_{13}a_{31}^3-b_{13}b_{31}^3=a_{04}a_{31}^5-b_{04}b_{31}^5=0. \tag{1.9}$$

Some linearizability conditions for system (1.8) with (1.9) are found in Section 3. Finally, returning our results on complex systems to real ones, in Section 4 we give some conditions of isochronous center for the quartic-like real system

$$\begin{cases}\dot{x}=-y+\big(x^2+y^2\big)^d\big(C_{40}x^4+C_{22}x^2y^2+C_{04}y^4\big),\\ \dot{y}=x+\big(x^2+y^2\big)^d\big(C_{31}x^3y+C_{13}xy^3\big),\end{cases} \tag{1.10}$$

where $x,y,C_{ij}\in\mathbb{R}$, $d\geqslant 0$ is a real number and the coefficients satisfy $C_{04}-C_{13}-C_{22}+C_{31}+C_{40}=0$.

2. Preliminaries and reduction to polynomial systems

In this section we construct a transformation to change polynomial-like system (1.6) into a polynomial one and find the relation between their linearizability problems. Then we show that the time-reversibility of system (1.6) is independent of d. We briefly describe the method of Darboux linearization for polynomial systems, which will be used in Section 3.

Since P_n and Q_n given in (1.6) are both homogeneous polynomials of degree n, let us assume

$$P_n(Z,W) := \sum_{i=0}^{n} a_{n-i,i} Z^{n-i} W^i, \qquad Q_n(Z,W) := \sum_{i=0}^{n} b_{n-i,i} Z^i W^{n-i}.$$

Theorem 2.1. *Polynomial-like system* (1.6) *is linearizable if and only if the analytic system*

$$\begin{cases} \dot U = U + \dfrac{d+n}{n+1} UV P_n(U,V) - \dfrac{d-1}{n+1} U^2 Q_n(U,V), \\ \dot V = -V - \dfrac{d+n}{n+1} UV Q_n(U,V) + \dfrac{d-1}{n+1} V^2 P_n(U,V), \end{cases} \tag{2.1}$$

a polynomial system with homogeneous nonlinearities of degree $n+2$, where $UV \neq 0$ or $(U,V)=(0,0)$, is linearizable.

Proof. Consider the transformation $(Z,W) \mapsto (U,V)$, where

$$(U,V) = \begin{cases} (Z^{\frac{d+n}{n+1}} W^{\frac{d-1}{n+1}}, Z^{\frac{d-1}{n+1}} W^{\frac{d+n}{n+1}}), & \text{if } ZW \neq 0, \\ (0,0), & \text{if } (Z,W)=(0,0). \end{cases} \tag{2.2}$$

It changes the polynomial-like system (1.6) into the polynomial system (2.1) with the restriction that either $UV \neq 0$ or $(U,V)=(0,0)$.

Assume that system (1.6) is linearizable, that is, there exists a near-identity transformation (1.7) analytic in the region Ξ such that (1.6) is linearized to the form $\dot X = X$, $\dot Y = -Y$ near the origin. From (2.2) we get

$$(Z,W) = \begin{cases} (U^{\frac{d+n}{n+2d-1}} V^{\frac{1-d}{n+2d-1}}, U^{\frac{1-d}{n+2d-1}} V^{\frac{d+n}{n+2d-1}}), & \text{if } UV \neq 0, \\ (0,0), & \text{if } (U,V)=(0,0). \end{cases}$$

Thus, both X and Y are also analytic functions of U and V near the origin in the region Ξ, i.e., $\{(U,V)\in\mathbb{C}^2\colon UV \neq 0\}$. Let

$$x = X^{\frac{d+n}{n+1}} Y^{\frac{d-1}{n+1}}, \qquad y = X^{\frac{d-1}{n+1}} Y^{\frac{d+n}{n+1}}. \tag{2.3}$$

It is easy to check that both x and y are also analytic functions of U and V and can be presented as $x = U + o(|(U,V)|)$, $y = V + o(|(U,V)|)$ near the origin in Ξ. Moreover, replacing with the transformation, one check the following

$$\begin{cases} \dot x = \dfrac{d+n}{n+1} X^{\frac{d+n}{n+1}-1} Y^{\frac{d-1}{n+1}} X + \dfrac{d-1}{n+1} X^{\frac{d+n}{n+1}} Y^{\frac{d-1}{n+1}-1}(-Y) = x, \\ \dot y = \dfrac{d-1}{n+1} X^{\frac{d-1}{n+1}-1} Y^{\frac{d+n}{n+1}} X + \dfrac{d+n}{n+1} X^{\frac{d-1}{n+1}} Y^{\frac{d+n}{n+1}-1}(-Y) = -y, \end{cases}$$

that is, system (2.1) is linearizable by a near-identity transformation which is analytic near the origin in Ξ.

Using the same argument, we can also prove that system (1.6) is linearizable if system (2.1) is linearizable. $\square$

We remark that in our Theorem 2.1 and the whole paper we discuss the linearizability with the restriction to Ξ. Actually, for a general complex polynomial system we may consider the linearizability problem without the restriction. However, if the considered complex polynomial system is originated from a real polynomial system of center, the result of linearizability with the restriction to Ξ is sufficient because the condition of Ξ holds naturally for the conjugacy, as indicated in the Introduction. In our paper we consider the polynomial-like system (1.6), for which the obtained transformation (1.7) may not be analytic without the restriction to Ξ. In deed, no matter whether we can find an analytic transformation in the linearization, we only need a transformation (1.7) which is analytic in Ξ. That is the essential difference from the linearization for complex polynomial systems.

For a real system, *time-reversibility* means that the system is invariant under reflexion with respect to a line passing through the origin and a change in the direction of time. This notion can be generalized to complex systems [18,26,28,29]. In general, a dynamical system described by the ordinary differential equation

$$\frac{d\mathbf{z}}{dt} = \mathcal{F}(\mathbf{z}), \quad \mathbf{z} \in \Omega,$$

where $\mathcal{F}: \Omega \to T\Omega$ is a vector field on a manifold Ω and $T\Omega$ is the tangent bundle, is said to be *time-reversible* (see e.g. [13]) if there is an invertible map $\Upsilon: \Omega \to \Omega$ such that

$$\frac{d\Upsilon(\mathbf{z})}{dt} = -F\big(\Upsilon(\mathbf{z})\big).$$

We will deal with a simple case of time-reversible symmetries, in which the system is 2-dimensional (i.e., $\mathbf{z} = (Z, W) \in \mathbb{C}^2$) and the map Υ is given by

$$U = \alpha W, \qquad V = \alpha^{-1} Z, \tag{2.4}$$

where $\alpha \in \mathbb{C} \setminus \{0\}$. It is easy to see that system (1.2) is time-reversible (with respect to the map (2.4)) if and only if there exists some α such that

$$\widetilde{P}(Z, W) \equiv -\alpha \widetilde{Q}\left(\alpha W, \alpha^{-1} Z\right) \tag{2.5}$$

(see e.g. [26,28] for details). If a complex system (1.2) is a time-reversible (with respect to map (2.4)) and such that the first and second equations are complex conjugate, then setting $Z = x + iy$ we obtain from (1.2) a real system of form (1.1) such that the vector field of the real system is symmetric with respect to a line passing through the origin. Time-reversible system (1.2) is integrable [18,26,28] and has an analytic first integral

$$\Phi(Z, W) = ZW + o\big(\big|(Z, W)\big|^2\big). \tag{2.6}$$

Proposition 2.1. *The time-reversibility of system* (1.6) *is independent of the value of* d.

Proof. Using (2.5), we easily check that system (1.2) is time-reversible if there is some α satisfying

$$P(Z, W) \equiv -\alpha Q\left(\alpha W, \alpha^{-1} Z\right). \tag{2.7}$$

It yields that system (1.6) is time-reversible if and only if there is a nonzero complex number α such that $P_n(Z, W) \equiv \alpha Q_n(\alpha W, \alpha^{-1} Z)$. The latter equation is independent of the value of d. This completes the proof. $\square$

Since our strategy is to reduce a non-polynomial problem to a polynomial one, in the remainder of this section we present some known methods to prove linearizability of system (1.2), where both P and Q are polynomials. Let f be some function and k be a polynomial satisfying

$$\frac{\partial f}{\partial x}\widetilde{P} + \frac{\partial f}{\partial y}\widetilde{Q} = fk.$$

The function f is called the *Darboux function* and k is called the *cofactor* of f. By the Darboux linearization method [23], the first equation of system (1.2) can be linearized to $\dot{X} = X$ by the institution $X = f_1 f_3^{\alpha_3} \cdots f_m^{\alpha_m}$ if

$$k_1 + \alpha_3 k_3 + \cdots + \alpha_m k_m \equiv 1 \tag{2.8}$$

holds for some $\alpha_3, \ldots, \alpha_m$; the second equation of system (1.2) can be linearized to $\dot{Y} = -Y$ by the institution $Y = f_2 f_3^{\beta_3} \cdots f_m^{\beta_m}$ if

$$k_2 + \beta_3 k_3 + \cdots + \beta_m k_m \equiv -1 \tag{2.9}$$

holds for some $\beta_3, \ldots, \beta_m$, where f_i's are Darboux functions of system (1.2) with cofactors k_i's and

$$f_1 = Z + o\big(\big|(Z, W)\big|\big), \qquad f_2 = W + o\big(\big|(Z, W)\big|\big), \tag{2.10}$$

$f_j = 1 + O(|(Z, W)|)$, $j = 3, \ldots, m$. If only one of (2.8) and (2.9) holds, the following result is used often.

Lemma 2.1. *(See [6].) Assume that system* (1.2) *has a first integral* (2.6)*. The second* (*respectively first*) *equation of* (1.2) *can be linearized by the substitution* $Y = \Phi(Z, W)/X(Z, W)$ (*respectively* $X = \Phi(Z, W)/Y(Z, W)$) *if the first* (*respectively second*) *equation of* (1.2) *can be linearized by* $X = X(Z, W)$ (*respectively* $Y = Y(Z, W)$).

Sometimes when we are not able to find sufficiently many Darboux functions to construct a Darboux linearization, the following observation is helpful.

Lemma 2.2. *(See [4].) Assume that system* (1.2) *has a first integral* (2.6) *and Darboux functions* f_1 *and* f_2 *given in* (2.10)*. Then system* (1.2) *has a Darboux function* $f_3 = 1 + O(|(Z, W)|)$ *with the cofactor* $-k_1 - k_2$.

3. Linearizability conditions for system (1.8)

In this section we use the transformation (2.2) to find linearizability conditions of time-reversible quartic-like system (1.8). Condition (1.9) is the time-reversibility condition for system (1.8) with $d = 0$ (see [4,28]). By Proposition 2.1, (1.9) is also the time-reversibility condition for (1.8) with any positive d.

Applying the transformation (2.2) with $n = 4$, we can change system (1.8) into

$$\begin{cases} \dot U = U - \dfrac{d-1}{5}\big(b_{04}U^6 + b_{13}U^5V + b_{22}U^4V^2 + b_{31}U^3V^3\big) \\ \qquad + \dfrac{d+4}{5}\big(a_{31}U^4V^2 + a_{22}U^3V^3 + a_{13}U^2V^4 + a_{04}UV^5\big), \\ \dot V = -V - \dfrac{d+4}{5}\big(b_{04}U^5V + b_{13}U^4V^2 + b_{22}U^3V^3 + b_{31}U^2V^4\big) \\ \qquad + \dfrac{d-1}{5}\big(a_{31}U^3V^3 + a_{22}U^2V^4 + a_{13}UV^5 + a_{04}V^6\big). \end{cases} \tag{3.1}$$

By Theorem 2.1 some linearizability conditions of system (1.8) with conditions (1.9) can be obtained from linearizability conditions of (3.1) with restriction (1.9). In the sequel we discuss in the four cases:

I. $a_{31}b_{31} \neq 0$; II. $a_{31} \neq 0$, $b_{31} = 0$; III. $a_{31} = 0$, $b_{31} \neq 0$; IV. $a_{31} = b_{31} = 0$.

We remind that we are assuming that $d \geqslant 0$. In case I we have the following result.

Theorem 3.1. *System* (1.8) *with* (1.9) *and* $a_{31}b_{31} \neq 0$ *is linearizable if one of the following four conditions holds*:

(1) $a_{04} = a_{13} = da_{22} + 2a_{22} - db_{31} - b_{31} = 0$;
(2) $a_{04} = a_{13} = a_{22} + b_{31} = 0$;
(3) $a_{04} = a_{22} = d = a_{31}a_{13} + b_{31}^2 = 0$;
(4) $d = 1069a_{04}a_{31}^2 - 750b_{31}^3 = 1069a_{13}a_{31} + 720b_{31}^2 = 2138a_{22} + 1331b_{31} = 0$.

Proof. When $a_{31}b_{31} \neq 0$, system (3.1) with (1.9) can be transformed into the system

$$\begin{cases} \dot X = X - \dfrac{d-1}{5}\big(A_{04}X^6 + A_{13}X^5Y + A_{22}X^4Y^2 + X^3Y^3\big) + \dfrac{d+4}{5}\big(X^4Y^2 + A_{22}X^3Y^3 + A_{13}X^2Y^4 + A_{04}XY^5\big), \\ \dot Y = -Y - \dfrac{d+4}{5}\big(A_{04}X^5Y + A_{13}X^4Y^2 + A_{22}X^3Y^3 + X^2Y^4\big) + \dfrac{d-1}{5}\big(X^3Y^3 + A_{22}X^2Y^4 + A_{13}XY^5 + A_{04}Y^6\big) \end{cases} \tag{3.2}$$

by the substitution

$$X = \alpha U, \qquad Y = \frac{b_{31}}{a_{31}}\alpha V,$$

where $\alpha = a_{31}^{3/5}b_{31}^{-2/5}$ and

$$A_{22} = a_{22}/b_{31}, \qquad A_{13} = a_{13}a_{31}/b_{31}^2, \qquad A_{04} = a_{04}a_{31}^2/b_{31}^3. \tag{3.3}$$

We compute the first 25 pairs of linearizability quantities [17,28] of system (3.2) and obtain that

$$j_k = i_k, \qquad i_{5k-h} = 0, \quad h = 1, \dots, 4,\ k = 1, \dots,$$
$$i_5 = 15 - 12A_{04}^2 - 15A_{13}^2 - 15A_{22} - 30A_{22}^2 + 15d - 3A_{04}^2d - 5A_{13}^2d - 15A_{22}^2d,$$

up to a constant multiplier. The polynomials i_{10}, i_{15}, i_{20} and i_{25} have 89, 424, 1215 and 2729 terms, respectively, so we do not present these polynomials here but the reader can compute them with any computer algebra system (e.g., *Mathematica*, *Maple*, etc.).

Using the routine *minAssChar* or *minAssGTZ* [7] of *Singular* [8] we find the irreducible decomposition

$$V\big(\langle i_5, i_{10}, \dots, i_{25}\rangle\big) = \bigcup_{k=1}^{9} \Lambda_k, \tag{3.4}$$

where $V(\langle i_5, i_{10}, \dots, i_{25}\rangle)$ is the variety of the ideal generated by $i_5, i_{10}, \dots, i_{25}$ and

$$
\begin{aligned}
&\Lambda_1 = \{(d, A_{04}, A_{13}, A_{22}) \in \mathbb{C}^4\colon\ A_{04} = A_{13} = A_{22}d + 2A_{22} - d - 1 = 0\},\\
&\Lambda_2 = \{(d, A_{04}, A_{13}, A_{22}) \in \mathbb{C}^4\colon\ A_{04} = A_{13} = A_{22} + 1 = 0\},\\
&\Lambda_3 = \{(d, A_{04}, A_{13}, A_{22}) \in \mathbb{C}^4\colon\ A_{04} = A_{13} + 1 = A_{22} = d = 0\},\\
&\Lambda_4 = \{(d, A_{04}, A_{13}, A_{22}) \in \mathbb{C}^4\colon\ 1069A_{04} - 750 = 1069A_{13} + 720 = 2138A_{22} + 1331 = d = 0\},\\
&\Lambda_5 = \{(d, A_{04}, A_{13}, A_{22}) \in \mathbb{C}^4\colon\ A_{04} = A_{22} - 2 = d + 3 = 0\},\\
&\Lambda_6 = \{(d, A_{04}, A_{13}, A_{22}) \in \mathbb{C}^4\colon\ A_{04} = A_{13} + 1 = 3A_{22} + 4 = d + 2 = 0\},\\
&\Lambda_7 = \{(d, A_{04}, A_{13}, A_{22}) \in \mathbb{C}^4\colon\ A_{04} = 5A_{13} + 3 = 5A_{22} + 2 = d + 2 = 0\},\\
&\Lambda_8 = \{(d, A_{04}, A_{13}, A_{22}) \in \mathbb{C}^4\colon\ A_{04} = 7A_{13} - 3 = 7A_{22} + 6 = d + 1 = 0\},\\
&\Lambda_9 = \{(d, A_{04}, A_{13}, A_{22}) \in \mathbb{C}^4\colon\ A_{13} = 2A_{22} - 3 = d + 4 = 0\}.
\end{aligned}
$$

Thus, the necessary condition for linearizability of a system (3.2) is that $(d, A_{04}, A_{13}, A_{22})$ is in one of the conditions Λ_k's. On the other hand, we only have to prove the sufficiency of Λ_k, $k = 1, \ldots, 4$, because $d \geqslant 0$. From (3.3) we see that $(d, A_{04}, A_{13}, A_{22}) \in \Lambda_1$ (respectively Λ_2, Λ_3, Λ_4) is equivalent to condition (1) (respectively (2), (3), (4)) presented in the theorem.

We now prove the sufficiency. When condition (1) of the theorem is satisfied, that is, $(d, A_{04}, A_{13}, A_{22}) \in \Lambda_1$, system (3.2) is

$$
\begin{cases}
\dot{X} = X + \dfrac{3(2d+3)}{5(d+2)}X^4Y^2 + \dfrac{2(2d+3)}{5(d+2)}X^3Y^3,\\
\dot{Y} = -Y - \dfrac{2(2d+3)}{5(d+2)}X^3Y^3 - \dfrac{3(2d+3)}{5(d+2)}X^2Y^4.
\end{cases}
\tag{3.5}
$$

It has two Darboux functions

$$
f_1 = 1 + \frac{2d+3}{d+2}X^2Y^3, \qquad f_2 = 1 + \frac{2d+3}{d+2}X^3Y^2,
$$

which yield the substitution $Z = Xf_1^{2/5}f_2^{-3/5}$ and $W = Yf_1^{-3/5}f_2^{2/5}$ linearizing (3.5) to $\dot{Z} = Z$, $\dot{W} = -W$. Thus, condition (1) in the theorem is sufficient for linearizability of system (1.8) with (1.9) and $a_{31}b_{31} \neq 0$.

When condition (2) of the theorem holds, i.e., $(d, A_{04}, A_{13}, A_{22}) \in \Lambda_2$, system (3.2) is

$$
\begin{cases}
\dot{X} = X + \dfrac{2d+3}{5}X^4Y^2 - \dfrac{2d+3}{5}X^3Y^3,\\
\dot{Y} = -Y + \dfrac{2d+3}{5}X^3Y^3 - \dfrac{2d+3}{5}X^2Y^4.
\end{cases}
\tag{3.6}
$$

It has the Darboux function $f_1 = 1 + (2d+3)X^3Y^2 + (2d+3)X^2Y^3$, which provides the linearizing substitution $Z = Xf_1^{-1/5}$ and $W = Yf_1^{-1/5}$. Thus, system (1.8) with (1.9) and $a_{31}b_{31} \neq 0$ is linearizable if condition (2) in the theorem is satisfied.

When condition (3) of the theorem is satisfied, system (1.8) with (1.9) and $a_{31}b_{31} \neq 0$ is written as

$$
\begin{cases}
\dot{Z} = Z + a_{31}Z^3W + a_{13}ZW^3,\\
\dot{W} = -W - b_{13}Z^3W - b_{31}ZW^3,
\end{cases}
\tag{3.7}
$$

where $a_{31}a_{13} + b_{31}^2 = 0$ and $b_{31}b_{13} + a_{31}^2 = 0$. The linearizability of system (3.7) is proved in Theorem 1 of [4] (case 3).

When condition (4) of the theorem is satisfied, system (1.8) with (1.9) and $a_{31}b_{31} \neq 0$ is written as

$$
\begin{cases}
\dot{Z} = Z + a_{31}Z^3W - \dfrac{1331b_{31}}{2138}Z^2W^2 - \dfrac{720b_{31}^2}{1069a_{31}}ZW^3 + \dfrac{750b_{31}^3}{1069a_{31}^2}W^4,\\
\dot{W} = -W - \dfrac{750a_{31}^3}{1069b_{31}^2}Z^4 + \dfrac{720a_{31}^2}{1069b_{31}}Z^3W + \dfrac{1331a_{31}}{2138}Z^2W^2 - b_{31}ZW^3.
\end{cases}
\tag{3.8}
$$

By the substitution

$$
U = \gamma Z, \qquad V = \frac{b_{31}\gamma}{a_{31}}W,
$$

where γ satisfies $1069b_{31}\gamma^3 + 750a_{31}^2 = 0$, system (3.8) is changed into

$$
\begin{cases}
\dot{U} = U - \dfrac{1069}{750}U^3V + \dfrac{1331}{1500}U^2V^2 + \dfrac{24}{25}UV^3 - V^4,\\
\dot{V} = -V + U^4 - \dfrac{24}{25}U^3V - \dfrac{1331}{1500}U^2V^2 + \dfrac{1069}{750}UV^3.
\end{cases}
\tag{3.9}
$$

This system satisfies the fifth condition of Theorem 3 of [4], where the linearizability is proved. Therefore, system (1.8) with (1.9) and $a_{31}b_{31} \neq 0$ is linearizable if condition (4) of the theorem is satisfied. The proof of the theorem is completed. □

In cases II and III, the system is linearizable, that is, we have the following statement.

Theorem 3.2. *System* (1.8) *with* (1.9) *is linearizable if one of the following two conditions holds*:

(1) $a_{31} \neq 0$, $b_{31} = 0$;
(2) $a_{31} = 0$, $b_{31} \neq 0$.

Proof. If $a_{31} \neq 0$ and $b_{31} = 0$, system (3.1) with (1.9) is written as

$$\begin{cases} \dot{U} = U - \dfrac{d-1}{5}\big(b_{04}U^6 + b_{13}U^5V + b_{22}U^4V^2\big) + \dfrac{d+4}{5}a_{31}U^4V^2, \\ \dot{V} = -V - \dfrac{d+4}{5}\big(b_{04}U^5V + b_{13}U^4V^2 + b_{22}U^3V^3\big) + \dfrac{d-1}{5}a_{31}U^3V^3. \end{cases} \tag{3.10}$$

We claim that the substitution

$$X = f(U,V) = \sum_{k=1}^{+\infty} f_k(V)U^k = \sum_{k=1}^{+\infty}\left(\sum_{i=0}^{k-1} a_i^{(k)}V^i\right)U^k \tag{3.11}$$

(that is, $f_k(V)$ is a polynomial of degree at most $k-1$) linearizes the first equation of system (3.10) to $\dot{X} = X$, where for $i \leqslant k-2$,

$$\begin{aligned} a_i^{(k)} = {} & \frac{1}{5(i+1-k)}\big((k-5)(1-d-id-4i)b_{04}a_i^{(k-5)} + (k-4)(5-id-4i)b_{13}a_{i-1}^{(k-4)} \\ & + (k-3)\big((id-d-i+6)a_{31} + (d-id-4i+9)b_{22}\big)a_{i-2}^{(k-3)}\big), \end{aligned} \tag{3.12}$$

$a_{j-1}^{(j)} = 1$ and $a_i^{(j)} = 0$ if $i < 0$ or $j \leqslant 0$ or $i > j-1$. Obviously, $f(U,V)$ is of form $U + o(|(U,V)|)$. In fact, in order to linearize the first equation of system (3.10) by $f(U,V)$ given in (3.11), we need only to prove that the equation

$$\begin{aligned} & 5(k-1)f_k - 5Vf_k' + (k-3)\big((d+4)a_{31} - (d-1)b_{22}\big)V^2 f_{k-3} + (k-3)\big((d-1)a_{31} - (d+4)b_{22}\big)V^3 f_{k-3}' \\ & \quad + (k-4)(1-d)b_{13}Vf_{k-4} - (k-4)(d+4)b_{13}V^2 f_{k-4}' + (k-5)(1-d)b_{04}f_{k-5} - (k-5)(d+4)b_{04}Vf_{k-5}' = 0 \end{aligned} \tag{3.13}$$

for $f_k(V)$ has a solution. One can check that $f_k(V)$ satisfying (3.12) is a solution of Eq. (3.13).

Using the same method, we see that there exists an analytic first integral of system (3.10)

$$\Phi(U,V) = \sum_{k=1}^{+\infty} h_k(V)U^k = \sum_{k=1}^{+\infty}\left(\sum_{i=0}^{k-1} b_i^{(k)}V^i\right)U^k, \tag{3.14}$$

where for $i \leqslant k-1$,

$$\begin{aligned} b_i^{(k)} = {} & \frac{1}{5(i-k)}\big((k-5)(1-d-id-4i)b_{04}b_i^{(k-5)} + (k-4)(5-id-4i)b_{13}b_{i-1}^{(k-4)} \\ & + (k-3)\big((id-d-i+6)a_{31} + (d-id-4i+9)b_{22}\big)b_{i-2}^{(k-3)}\big), \end{aligned} \tag{3.15}$$

$b_j^{(j)} = 1$ and $b_i^{(j)} = 0$ if $i < 0$ or $j \leqslant 0$ or $i > j$. Obviously, $\Phi(U,V)$ is of form $UV + o(|(U,V)|^2)$. Therefore, by Lemma 2.1 the second equation of system (3.10) can be linearized by the substitution $Y = \Phi(U,V)/f(U,V)$, where $f(U,V)$ and $\Phi(U,V)$ are given in (3.11) and (3.14), respectively.

When $a_{31} = 0$ and $b_{31} \neq 0$, system (3.1) with (1.9) is

$$\begin{cases} \dot{U} = U - \dfrac{d-1}{5}b_{31}U^3V^3 + \dfrac{d+4}{5}\big(a_{22}U^3V^3 + a_{13}U^2V^4 + a_{04}UV^5\big), \\ \dot{V} = -V - \dfrac{d+4}{5}b_{31}U^2V^4 + \dfrac{d-1}{5}\big(a_{22}U^2V^4 + a_{13}UV^5 + a_{04}V^6\big). \end{cases} \tag{3.16}$$

With the substitution $X = V$, $Y = U$ and $\tau = -T$, system (3.16) is changed into

$$\begin{cases} \dfrac{dX}{d\tau} = X - \dfrac{d-1}{5}\bigl(a_{04}X^6 + a_{13}X^5Y + a_{22}X^4Y^2\bigr) + \dfrac{d+4}{5}b_{31}X^4Y^2, \\ \dfrac{dY}{d\tau} = -Y - \dfrac{d+4}{5}\bigl(a_{04}X^5Y + a_{13}X^4Y^2 + a_{22}X^3Y^3\bigr) + \dfrac{d-1}{5}b_{31}X^3Y^3, \end{cases} \tag{3.17}$$

which is of form (3.10), and hence can be linearized to $dZ/d\tau = Z$, $dW/d\tau = -W$ by a transformation

$$Z = X + o\bigl(\bigl|(X,Y)\bigr|\bigr) = V + o\bigl(\bigl|(U,V)\bigr|\bigr), \qquad W = Y + o\bigl(\bigl|(X,Y)\bigr|\bigr) = U + o\bigl(\bigl|(U,V)\bigr|\bigr). \tag{3.18}$$

Therefore, system (3.16) is linearized by the transformation (3.18). □

The following theorem is devoted to case IV.

Theorem 3.3. *System* (1.8) *with* (1.9) *and* $a_{31} = b_{31} = 0$ *is linearizable if one of the following two conditions holds*:

(1) $a_{04} = a_{13} = a_{22} = 0$;
(2) $b_{04} = b_{13} = b_{22} = 0$.

Proof. When $a_{31} = b_{31} = 0$, system (3.1) is

$$\begin{cases} \dot U = U - \dfrac{d-1}{5}\bigl(b_{04}U^6 + b_{13}U^5V + b_{22}U^4V^2\bigr) + \dfrac{d+4}{5}\bigl(a_{22}U^3V^3 + a_{13}U^2V^4 + a_{04}UV^5\bigr), \\ \dot V = -V - \dfrac{d+4}{5}\bigl(b_{04}U^5V + b_{13}U^4V^2 + b_{22}U^3V^3\bigr) + \dfrac{d-1}{5}\bigl(a_{22}U^2V^4 + a_{13}UV^5 + a_{04}V^6\bigr) \end{cases} \tag{3.19}$$

and (1.9) becomes

$$a_{13}^5b_{04}^3 - a_{04}^3b_{13}^5 = a_{22}^3b_{13} - a_{13}b_{22}^3 = a_{22}^5b_{04} - a_{04}b_{22}^5 = 0. \tag{3.20}$$

We compute the first 25 pairs of linearizability quantities [17,28] of system (3.19) and obtain that

$$i_{5k-h} = j_{5k-h} = 0, \quad h = 1,\dots,4,\ k = 1,\dots,$$
$$i_5 = j_5 = -12a_{04}b_{04} - 15a_{13}b_{13} - 30a_{22}b_{22} - 3a_{04}b_{04}d - 5a_{13}b_{13}d - 15a_{22}b_{22}d,$$

up to a constant multiplier. The polynomials i_{10}, j_{10}, i_{15}, j_{15}, i_{20}, j_{20}, i_{25} and j_{25} have 48, 48, 192, 192, 584, 584, 1410 and 1410 terms, respectively, so we do not present these polynomials here but the reader can easily compute them with any available computer algebra system.

Using the routine *minAssChar* or *minAssGTZ* of *Singular* [8], we find the irreducible decomposition

$$V\bigl(\bigl\langle a_{13}^5b_{04}^3 - a_{04}^3b_{13}^5, a_{22}^3b_{13} - a_{13}b_{22}^3, a_{22}^5b_{04} - a_{04}b_{22}^5, i_5, j_5, \dots, i_{25}, j_{25}\bigr\rangle\bigr) = \bigcup_{k=1}^{5} \Gamma_k, \tag{3.21}$$

where

$$\Gamma_1 = \bigl\{(d, a_{04}, a_{13}, a_{22}, b_{04}, b_{13}, b_{22}) \in \mathbb{C}^7\colon\ a_{04} = a_{13} = a_{22} = 0\bigr\},$$
$$\Gamma_2 = \bigl\{(d, a_{04}, a_{13}, a_{22}, b_{04}, b_{13}, b_{22}) \in \mathbb{C}^7\colon\ b_{04} = b_{13} = b_{22} = 0\bigr\},$$
$$\Gamma_3 = \bigl\{(d, a_{04}, a_{13}, a_{22}, b_{04}, b_{13}, b_{22}) \in \mathbb{C}^7\colon\ d+4 = a_{13} = a_{22} = b_{13} = b_{22} = 0\bigr\},$$
$$\Gamma_4 = \bigl\{(d, a_{04}, a_{13}, a_{22}, b_{04}, b_{13}, b_{22}) \in \mathbb{C}^7\colon\ d+3 = a_{04} = a_{22} = b_{04} = b_{22} = 0\bigr\},$$
$$\Gamma_5 = \bigl\{(d, a_{04}, a_{13}, a_{22}, b_{04}, b_{13}, b_{22}) \in \mathbb{C}^7\colon\ d+2 = a_{04} = a_{13} = b_{04} = b_{13} = 0\bigr\}.$$

Recall that we only consider $d \geqslant 0$. Thus, system (3.19) with (3.20) can be linearizable only when condition (1) or (2) holds.

When condition (1) is satisfied, system (3.19) with (3.20) becomes

$$\begin{cases} \dot U = U - \dfrac{d-1}{5}\bigl(b_{04}U^6 + b_{13}U^5V + b_{22}U^4V^2\bigr), \\ \dot V = -V - \dfrac{d+4}{5}\bigl(b_{04}U^5V + b_{13}U^4V^2 + b_{22}U^3V^3\bigr). \end{cases} \tag{3.22}$$

It has Darboux functions $f_1 = U$ and $f_2 = V$ with the cofactors

$$k_1(U,V) = 1 - \frac{d-1}{5}\bigl(b_{04}U^5 + b_{13}U^4V + b_{22}U^3V^2\bigr),$$
$$k_2(U,V) = -1 - \frac{d+4}{5}\bigl(b_{04}U^5 + b_{13}U^4V + b_{22}U^3V^2\bigr),$$

respectively. By Lemma 2.2, system (3.22) has a Darboux function $f_3 = 1 + O(|(U,V)|)$ with the cofactor

$$k_3(U,V) = -\big(k_1(U,V) + k_2(U,V)\big) = \frac{2d+3}{5}\big(b_{04}U^5 + b_{13}U^4V + b_{22}U^3V^2\big),$$

which yields the substitution

$$X = Uf_3^{\frac{d-1}{2d+3}}, \qquad Y = Vf_3^{\frac{d+4}{2d+3}}$$

linearizing system (3.22) to $\dot X = X$, $\dot Y = -Y$.

When condition (2) is satisfied, system (3.19) with (3.20) is written as

$$\begin{cases} \dot U = U + \dfrac{d+4}{5}\big(a_{22}U^3V^3 + a_{13}U^2V^4 + a_{04}UV^5\big), \\ \dot V = -V + \dfrac{d-1}{5}\big(a_{22}U^2V^4 + a_{13}UV^5 + a_{04}V^6\big). \end{cases} \tag{3.23}$$

It admits Darboux functions $f_1 = U$ and $f_2 = V$ with the cofactors

$$k_1(U,V) = 1 + \frac{d+4}{5}\big(a_{22}U^2V^3 + a_{13}UV^4 + a_{04}V^5\big),$$
$$k_2(U,V) = -1 + \frac{d-1}{5}\big(a_{22}U^2V^3 + a_{13}UV^4 + a_{04}V^5\big),$$

respectively. By Lemma 2.2, system (3.23) has a Darboux function $f_3 = 1 + O(|(U,V)|)$ with the cofactor

$$k_3(U,V) = -\big(k_1(U,V) + k_2(U,V)\big) = -\frac{2d+3}{5}\big(a_{22}U^2V^3 + a_{13}UV^4 + a_{04}V^5\big),$$

which yields the substitution

$$X = Uf_3^{\frac{d+4}{2d+3}}, \qquad Y = Vf_3^{\frac{d-1}{2d+3}}$$

linearizing system (3.23). □

4. Concluding remarks

In this section we give two remarks to our results and some results obtained in [4,21]. We discuss the relation between our results and the isochronicity of quartic-like real system (1.10).

In [21] the authors consider the equation

$$\dot Z = Z + (Z\overline{Z})^{\frac{\widetilde d-4}{2}}\big(-\mathrm{i}AZ^3\overline{Z} - \mathrm{i}BZ^2\overline{Z}^2 - \mathrm{i}C\overline{Z}^4\big), \quad \widetilde d \geqslant 4 \text{ is even},$$

which can be written as system (1.8) where

$$d = \frac{\widetilde d - 4}{2}, \qquad a_{31} = -\mathrm{i}A, \qquad a_{22} = -\mathrm{i}B, \qquad a_{13} = 0, \qquad a_{04} = -\mathrm{i}C, \qquad b_{ij} = \overline{a_{ij}}, \tag{4.1}$$

and W is the conjugacy $\overline{Z}$ of Z. It is proved in [21, Theorem 1] that system (1.8) with condition (4.1) is linearizable if and only if

$$C = B - \overline{A} = 0 \quad \text{or} \quad C = \widetilde d B - (\widetilde d - 2)\overline{A} = 0, \tag{4.2}$$

that is,

$$C = B - \overline{A} = 0 \quad \text{or} \quad C = (d+2)B - (d+1)\overline{A} = 0. \tag{4.3}$$

However, by Theorems 3.1, 3.2 and 3.3 we obtain the following results.

Remark 4.1. System (1.8) with (4.1) is linearizable if

$$C = B - \overline{A} = 0 \quad \text{or} \quad C = (d+2)B + (d+1)\overline{A} = 0. \tag{4.4}$$

Note that the second conditions of (4.4) and (4.3) are different from each other. We guess the second condition of (4.3) is not correct, that is, the second condition of (4.2) should be $C = \widetilde{d}B - (2 - \widetilde{d})\overline{A} = 0$, which means that there is a misprint in condition (b.2) in Theorem 1 of [21]. The proof of Theorem 1 of [21] on page 3126 also provides an evidence for our guess.

When $d = 0$, quartic-like system (1.8) is actually the quartic system

$$\begin{cases} \dot{Z} = Z + a_{31}Z^3W + a_{22}Z^2W^2 + a_{13}ZW^3 + a_{04}W^4, \\ \dot{W} = -W - b_{04}Z^4 - b_{13}Z^3W - b_{22}Z^2W^2 - b_{31}ZW^3, \end{cases} \tag{4.5}$$

where the coefficients satisfy the reversibility condition (1.9). Restricting our results of Theorems 3.1, 3.2 and 3.3 to system (4.5), we obtain some conditions for (4.5) to be linearizable. On the other hand, the authors in [4] give linearizability conditions for general time-reversible quartic systems with homogeneous nonlinearities. By Theorems 1, 2 and 3 of [4], system (4.5) is linearizable if and only if it can be transformed into one of the following eight systems

$$(S_1):\quad \begin{cases} \dot{Z} = Z + a_{31}Z^3W, \\ \dot{W} = -W - b_{13}Z^3W - b_{22}Z^2W^2; \end{cases} \qquad (S_2):\quad \begin{cases} \dot{Z} = Z + a_{22}Z^2W^2 + a_{13}ZW^3, \\ \dot{W} = -W - b_{31}ZW^3; \end{cases}$$

$$(S_3):\quad \begin{cases} \dot{Z} = Z + a_{31}Z^3W + a_{13}ZW^3, \\ \dot{W} = -W - b_{13}Z^3W - b_{31}ZW^3, \end{cases}$$

where $a_{13}b_{13} - a_{31}b_{31} = a_{31}a_{13} + b_{31}^2 = a_{31}^2 + b_{13}b_{31} = 0$;

$$(S_4):\quad \begin{cases} \dot{Z} = Z + a_{31}Z^3W - b_{31}Z^2W^2, \\ \dot{W} = -W + a_{31}Z^2W^2 - b_{31}ZW^3; \end{cases} \qquad (S_5):\quad \begin{cases} \dot{Z} = Z + a_{31}Z^3W + \dfrac{b_{31}}{2}Z^2W^2, \\ \dot{W} = -W - \dfrac{a_{31}}{2}Z^2W^2 - b_{31}ZW^3; \end{cases}$$

$$(S_6):\quad \begin{cases} \dot{Z} = Z + a_{22}Z^2W^2 + a_{13}ZW^3 + W^4, \\ \dot{W} = -W - b_{31}ZW^3; \end{cases} \qquad (S_7):\quad \begin{cases} \dot{Z} = Z + a_{31}Z^3W, \\ \dot{W} = -W - Z^4 - b_{13}Z^3W - b_{22}Z^2W^2; \end{cases}$$

$$(S_8):\quad \begin{cases} \dot{Z} = Z + \dfrac{1069}{750}Z^3W - \dfrac{1331}{1500}Z^2W^2 - \dfrac{24}{25}ZW^3 + W^4, \\ \dot{W} = -W - Z^4 + \dfrac{24}{25}Z^3W + \dfrac{1331}{1500}Z^2W^2 - \dfrac{1069}{750}ZW^3 \end{cases}$$

by a substitution $Z \to \alpha Z$, $W \to \beta W$, where $\alpha, \beta \in \mathbb{C} \setminus \{0\}$.

Remark 4.2. Systems (S_1)–(S_8) are equivalent to ones determined by conditions of Theorems 3.1, 3.2 and 3.3 with $d = 0$.

In more details, system (S_1) satisfies condition (1) of Theorem 3.2 when $a_{31} \neq 0$, condition (1) of Theorem 3.3 when $a_{31} = 0$; system (S_2) satisfies condition (2) of Theorem 3.2 when $b_{31} \neq 0$, condition (2) of Theorem 3.3 when $b_{31} = 0$; system (S_3) satisfies condition (3) of Theorem 3.1 when $a_{31}b_{31} \neq 0$, condition (1) or (2) of Theorem 3.3 when $a_{31}b_{31} = 0$; system (S_4) satisfies condition (2) of Theorem 3.1 when $a_{31}b_{31} \neq 0$, condition (1) or (2) of Theorem 3.2 or condition (1) of Theorem 3.3 when $a_{31}b_{31} = 0$; system (S_5) satisfies condition (1) of Theorem 3.1 when $a_{31}b_{31} \neq 0$, condition (1) or (2) of Theorem 3.2 or condition (1) of Theorem 3.3 when $a_{31}b_{31} = 0$; system (S_6) satisfies condition (2) of Theorem 3.2 when $b_{31} \neq 0$, condition (2) of Theorem 3.3 when $b_{31} = 0$; system (S_7) satisfies condition (1) of Theorem 3.2 when $a_{31} \neq 0$, condition (1) of Theorem 3.3 when $a_{31} = 0$; system (S_8) satisfies condition (4) of Theorem 3.1. On the other hand, the system satisfying condition (1) of Theorem 3.1 is (S_5); the system satisfying condition (2) of Theorem 3.1 is (S_4); the system satisfying condition (3) of Theorem 3.1 is (S_3); the system satisfying condition (4) of Theorem 3.1 can be reduced to (S_8); the system satisfying condition (1) of Theorem 3.2 can be reduced to (S_7) when $b_{04} \neq 0$ and is (S_1) when $b_{04} = 0$; the system satisfying condition (2) of Theorem 3.2 can be reduced to (S_6) when $a_{04} \neq 0$ and is (S_2) when $a_{04} = 0$; the system satisfying condition (1) of Theorem 3.3 can be reduced to (S_7) when $b_{04} \neq 0$ and is (S_1) when $b_{04} = 0$; the system satisfying condition (2) of Theorem 3.3 can be reduced to (S_6) when $a_{04} \neq 0$ and is (S_2) when $a_{04} = 0$.

Complexifying real system (1.10) by $Z = x + \mathrm{i}y$, $W = x - \mathrm{i}y$ and $T = \mathrm{i}t$, we obtain complex system (1.8) with

$$\begin{aligned} &b_{ij} = -a_{ij}, \qquad a_{31} = \mathrm{i}(2C_{04} - 2C_{40} - C_{13} - C_{31})/8, \qquad a_{22} = -\mathrm{i}(3C_{04} + C_{22} + 3C_{40})/8, \\ &a_{13} = \mathrm{i}(2C_{04} - 2C_{40} + C_{13} + C_{31})/8, \qquad a_{04} = -\mathrm{i}(C_{04} - C_{22} + C_{40} + C_{13} - C_{31})/16. \end{aligned} \tag{4.6}$$

From Theorems 3.1, 3.2, 3.3 and (4.6) we obtain the following result.

Corollary 4.1. *System* (1.10) *with* $2C_{04} - 2C_{40} - C_{13} - C_{31} \neq 0$ *has an isochronous center at the origin if one of the following conditions holds*:

(1) $C_{04} - C_{22} + C_{40} + C_{13} - C_{31} = 2C_{04} - 2C_{40} + C_{13} + C_{31} = (d+1)(2C_{04} - 2C_{40} - C_{13} - C_{31}) - (d+2)(3C_{04} + C_{22} + 3C_{40}) = 0$;

(2) $C_{04} - C_{22} + C_{40} + C_{13} - C_{31} = 2C_{04} - 2C_{40} + C_{13} + C_{31} = 5C_{04} + C_{22} + C_{40} - C_{13} - C_{31} = 0$;
(3) $d = C_{04} - C_{22} + C_{40} + C_{13} - C_{31} = 3C_{04} + C_{22} + 3C_{40} = C_{04} - C_{40} = 0$;
(4) $d = 1931C_{04} + 1069C_{22} - 4069C_{40} - 2569C_{13} - 431C_{31} = 3578C_{04} - 3578C_{40} + 349C_{13} + 349C_{31} = 9076C_{04} + 2138C_{22} + 3752C_{40} - 1331C_{13} - 1331C_{31} = 0$.

We check that conditions (1)–(4) of Corollary 4.1 satisfy conditions (6), (5), (1) of Theorem 4 in [4] and condition (4) of Theorem 5 in [4], respectively. This provides an additional check for the correctness of the obtained results.

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